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Uniform partitions of frames of exponentials into Riesz sequences

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ABSTRACT

The Feichtinger conjecture, if true, would have as a corollary that for each set $E \subset [0,1]$ and $\Lambda \subset \mathbb{Z}$, there is a partition $\Lambda_1, \ldots, \Lambda_N$ of $\mathbb{Z}$ such that for each $1 \leqslant i \leqslant N$, $\{\exp(2\pi i x\lambda)\colon \lambda \in \Lambda_i\}$ is a Riesz sequence. In this paper, sufficient conditions on sets $E \subset [0,1]$ and $\Lambda \subset \mathbb{R}$ are given so that $\{\exp(2\pi i x\lambda)1_E\colon \lambda \in \Lambda\}$ can be uniformly partitioned into Riesz sequences.

1. Introduction

A frame is a collection of elements $\{e_i\colon i \in I\}$ in a Hilbert space $\mathcal{H}$ such that there exist positive constants A and B such that for every $h \in \mathcal{H}$,

$$A\|h\|^2 \leqslant \sum_{i\in I} \left|\langle h, e_i\rangle\right|^2 \leqslant B\|h\|^2.$$

A frame $\{e_i\colon i \in I\}$ is *bounded* if

$$\inf_{i\in I} \|e_i\| > 0.$$

(Note that it is automatic that $\sup_{i\in I} \|e_i\| < \infty$.) A sequence $\{e_i\colon i \in I\}$ is said to be a Riesz sequence if it is a Riesz basis for its closed linear span, i.e., there exist $K_1, K_2 > 0$ such that for every finite family of scalars $\{a_i\colon i \in I\}$

$$K_1 \sum_{i\in I} |a_i|^2 \leqslant \left\| \sum_{i\in I} a_i e_i \right\|^2 \leqslant K_2 \sum_{i\in I} |a_i|^2.$$

Note that for Riesz basic sequences we always have

$$0 < \inf_{i\in I} \|e_i\| \leqslant \sup_{i\in I} \|e_i\| < \infty.$$

With this notation, one can state the Feichtinger conjecture:

Conjecture 1 *(Feichtinger). Every bounded frame can be written as the finite disjoint union of Riesz basic sequences.*

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The Feichtinger conjecture has been shown to be related to several famous open problems in analysis, and as such, is receiving a fair amount of recent interest [3,6,7,10]. Of particular interest is the special case of the Feichtinger conjecture for frames of exponentials, e.g. frames of the form $\{\exp(2\pi i\lambda x)1_E\colon \lambda \in \Lambda\}$, where E is measurable subset of $[0,1]$ with positive measure. The best positive result so far is in [1], where it is shown that in the case $\Lambda = \mathbb{Z}$, frames of exponentials always contain a Riesz sequence with positive Beurling density.

The most natural type of partition of a set indexed by the integers would be a uniform partition, so it is natural to ask which frames can be uniformly partitioned into Riesz sequences. We state this formally as a definition.

Definition 2. Let $\Lambda = \{\cdots < \lambda_{-1} < \lambda_0 < \lambda_1 < \cdots\} \subset \mathbb{R}$. We say that $\{e_{\lambda_k}\colon k \in \mathbb{Z}\}$ can be uniformly partitioned into Riesz sequences if there exists an N such that for $1 \leqslant J \leqslant N$, $\{e_{\lambda_{mN+J}}\colon m \in \mathbb{Z}\}$ is a Riesz sequence.

Gröchenig [10] showed that if a frame is intrinsically localized, than it can be uniformly partitioned into Riesz sequences. Bownik and the author [3] observed that if E contains an interval a.e., then $\{\exp(2\pi ixn)1_E\colon n \in \mathbb{Z}\}$ can be uniformly partitioned into Riesz sequences. Halpern, Kaftal and Weiss [11] showed that if $\phi \in L^\infty([0,1])$ is Riemann integrable, then the associated Laurent operator L_ϕ can be uniformly paved. Moreover, to date the primary negative evidence offered against the Feichtinger conjecture is two constructions of frames which can not be uniformly partitioned into Riesz sequences in a strong way, see [3, Theorem 4.13] and [11, Theorem 5.4(b)].

In this paper, we provide a sufficient condition on the pair (E,Λ), where $E \subset [0,1]$ and $\Lambda \subset \mathbb{R}$, such that $\{\exp(2\pi i\lambda x)1_E\colon \lambda \in \Lambda\}$ can be uniformly partitioned into Riesz sequences. Of particular interest is the application of this condition to the example considered in [2,11], which shows that, perhaps, the proposed counterexample to the paving conjecture in [11], which was proven not to be a counterexample in [2], was not optimally chosen. See Example 12 for details.

2. Beurling dimension

In this section, we recall some facts about the Beurling dimension of a subset of $\mathbb{R}^d$, though we will be concerned only with subsets of $\mathbb{R}$. For $h > 0$, we let Q denote the cube $[-1,1]^d$ and let Q_h be the dilation of Q by the factor of h:

$$Q_h = hQ = [-h,h]^d.$$

For any $x = (x_1,\dots,x_d) \in \mathbb{R}^d$ we let $Q_h(x)$ be the set Q_h translated in such a way so that it is "centered" at x, i.e.,

$$Q_h(x) = \prod_{i=1}^{d} [x_i - h, x_i + h].$$

Employing these notions we will first define a generalization of Beurling density.

Definition 3. Let $\Lambda \subset \mathbb{R}^d$ and $r > 0$. Then the *lower Beurling density of Λ with respect to r* is defined by

$$\mathcal{D}_r^-(\Lambda) = \liminf_{h\to\infty} \inf_{x\in\mathbb{R}^d} \frac{\#(\Lambda \cap Q_h(x))}{h^r},$$

and the *upper Beurling density of Λ with respect to r* is defined by

$$\mathcal{D}_r^+(\Lambda) = \limsup_{h\to\infty} \sup_{x\in\mathbb{R}^d} \frac{\#(\Lambda \cap Q_h(x))}{h^r}.$$

If $\mathcal{D}_r^-(\Lambda) = \mathcal{D}_r^+(\Lambda)$, then we say that Λ has *uniform Beurling density with respect to r* and denote this density by $\mathcal{D}_r(\Lambda)$.

With this definition in hand, we can define the upper and lower Beurling dimensions of subsets of $\mathbb{R}^d$.

Definition 4. Let $\Lambda \subset \mathbb{R}^d$. Then the *lower dimension of $\Lambda \subset \mathbb{R}^d$* is defined by

$$\dim^-(\Lambda) = \inf\{r > 0\colon \mathcal{D}_r^-(\Lambda) < \infty\}$$

and the *upper dimension of $\Lambda \subset \mathbb{R}^d$* is

$$\dim^+(\Lambda) = \sup\{r > 0\colon \mathcal{D}_r^+(\Lambda) > 0\}.$$

When these two quantities are equal, we refer to the *Beurling dimension* of Λ, and we denote it by $\dim(\Lambda)$.

We note here that the upper Beurling dimension is a base point independent version of the *upper mass dimension* considered in [4], see [8,9] for details. One result on Beurling dimension that we will use in the main section of this paper is the following [8,9].

Proposition 5. *Let $\Lambda \subset \mathbb{R}^d$.*

(i) *The following conditions are equivalent.*
  (a) $\mathcal{D}_d^+(\Lambda) < \infty$.
  (b) *There exists some $h > 0$ such that* $\sup_{x\in\mathbb{R}^d} \#(\Lambda \cap Q_h(x)) < \infty$.
  (c) *For all $h > 0$,* $\sup_{x\in\mathbb{R}^d} \#(\Lambda \cap Q_h(x)) < \infty$.
  (d) *Λ is relatively uniformly discrete.*
  (e) *For all $h > 0$,* $\sup_{x\in\mathbb{R}^d} \#\{\lambda \in \Lambda\colon x \in Q_h(\lambda)\} < \infty$.
(ii) *Also the following conditions are equivalent.*
  (a) $\mathcal{D}_d^-(\Lambda) > 0$.
  (b) *There exists some $h > 0$ such that* $\inf_{x\in\mathbb{R}^d} \#(\Lambda \cap Q_h(x)) > 0$.
  (c) *Λ contains a subsequence of positive uniform density.*
  (d) *There exists some $h > 0$ such that Λ is h-dense.*

3. Results

For E a measurable subset of $[0,1]$ and $\Lambda \subset \mathbb{R}$, we will say that (E, Λ) can be uniformly partitioned into Riesz sequences if the frame $\{\exp(2\pi i\lambda x)1_E\colon \lambda \in \Lambda\}$ can be uniformly partitioned into Riesz sequences, see Definition 2. Characterizing sets E such that $(E, \mathbb{Z})$ can be uniformly partitioned into Riesz sequences is related (but not equivalent) to characterizing the Laurent operators which can be uniformly paved, which was characterized in [11] as those Laurent operators whose symbol is Riemann integrable. In this article, we consider general sets $\Lambda \subset \mathbb{R}$.

Our main tool is a theorem due to Montgomery and Vaughan: [12, Theorem 1, Chapter 7], [13].

Theorem 6. *Suppose that $\lambda_1, \dots, \lambda_N$ are distinct real numbers, and suppose that $\delta > 0$ is chosen so that $|\lambda_n - \lambda_m| \geqslant \delta$ whenever $n \neq m$. Then, for any coefficients $a_1, \dots, a_N$, and any $T > 0$,*

$$(T - 1/\delta)\sum_{n=1}^N |a_n|^2 \leqslant \int_0^T \left|\sum_{n=1}^N a_n e^{2\pi i\lambda_n t}\right|^2 dt \leqslant (T + 1/\delta)\sum_{n=1}^N |a_n|^2. \tag{3.1}$$

We will also need two lemmas concerning partitioning subsets of $\mathbb{R}$ with finite upper dimension.

Lemma 7. *Let $\Lambda = \{\cdots < \lambda_{-1} < \lambda_0 < \lambda_1 < \cdots\} \subset \mathbb{R}$ such that $\dim^+(\Lambda) \leqslant 1$, and $K \in \mathbb{N}$. There exists $N \in \mathbb{N}$ such that whenever $|i - j| > N$, $|\lambda_i - \lambda_j| > K$.*

Proof. This is just a restatement of Proposition 5. □

For $\Lambda \subset \mathbb{R}$, $\alpha \geqslant 0$ and $r > 0$, we define $D^+_{\alpha,\Lambda}(r) = \sup\{\frac{\#(\Lambda\cap Q_r(x))}{r^\alpha}\colon\ x \in \mathbb{R}\}$.

Lemma 8. *Let $\Lambda = \{\cdots < \lambda_{-1} < \lambda_0 < \lambda_1 < \cdots\} \subset \mathbb{R}$ be such that $\dim^+(\Lambda) < \beta \leqslant 1$, and $\epsilon > 0$. There exist $N, R \in \mathbb{N}$ such that for each $1 \leqslant j \leqslant N$ and $r \geqslant R$, $D^+_{\beta,\Lambda_j(N)}(r) \leqslant 2R^{-\beta} + \epsilon$, where $\Lambda_j(N) = \{\lambda_{mN+j}\colon\ m \in \mathbb{Z}\}$.*

Proof. Choose R such that for $r \geqslant R$,

$$\sup_{x\in\mathbb{R}} \frac{\#(\Lambda \cap Q_r(x))}{r^\beta} \leqslant \sup_{x\in\mathbb{R}} \frac{\#(\Lambda \cap Q_R(x))}{R^\beta} + \epsilon < \infty.$$

Choose $N = \sup_{x\in\mathbb{R}} \#(\Lambda \cap Q_R(x))$. Now, fix $r \geqslant R$ and $1 \leqslant j \leqslant N$. Then,

$$\begin{aligned} D^+_{\beta,\Lambda_j(N)}(r) &= \sup_{x\in\mathbb{R}} \frac{\#(\Lambda_j(N) \cap Q_r(x))}{r^\beta} \\ &\leqslant \sup_{x\in\mathbb{R}} \frac{\#(\Lambda \cap Q_r(x))/N + 1}{r^\beta} \\ &\leqslant \sup_{x\in\mathbb{R}} \frac{\#(\Lambda \cap Q_R(x))}{R^\beta N} + \epsilon + R^{-\beta} \\ &\leqslant 2R^{-\beta} + \epsilon. \qquad \square \end{aligned}$$

Using Theorem 6, we obtain the following improvement to Lemma 5.1 in [5].

Lemma 9. *Suppose $\Lambda \subset \mathbb{R}$. If I is an interval contained in $[0,1]$, then for any sequence of numbers $\{a_\lambda\}_{\lambda\in\Lambda}$ in ℓ^2,*

$$\int_I \Big|\sum a_\lambda e^{2\pi i\lambda\xi}\Big|^2 \leqslant 2\ell(I) D^+_{0,\Lambda}\big(\ell(I)^{-1}\big). \tag{3.2}$$

Proof. First note that the interval $[0,T]$ in Theorem 6 can be replaced by any interval of length T, which we set to be $\ell(I)$. We can partition Λ into $D^+_{0,\Lambda}(\ell(I)^{-1})$ subsets Λ_i such that if λ_1,λ_2 are in Λ_i, then $|\lambda_1-\lambda_2|>\ell(I)^{-1}$. It follows that

$$\begin{aligned}\bigg(\int_I \Big|\sum a_\lambda e^{2\pi i\lambda\xi}\Big|^2\bigg)^{1/2} &\leqslant \sum_i\bigg(\int_I\Big|\sum_{\lambda\in\Lambda_i} a_\lambda e^{2\pi i\lambda\xi}\Big|^2\bigg)^{1/2}\\ &\leqslant \sum_i (2\ell(I))^{1/2}\bigg(\sum_{\lambda\in\Lambda_i}|a_\lambda|^2\bigg)^{1/2}\\ &\leqslant \big(2\ell(I)\big)^{1/2}\big(D^+_{0,\Lambda}\big(\ell(I)^{-1}\big)\big)^{1/2}\bigg(\sum_i\sum_{\lambda\in\Lambda_i}|a_\lambda|^2\bigg)^{1/2}\\ &= \big(2\ell(I)\big)^{1/2}\big(D^+_{0,\Lambda}\big(\ell(I)^{-1}\big)\big)^{1/2}\bigg(\sum_{\lambda\in\Lambda}|a_\lambda|^2\bigg)^{1/2},\end{aligned}$$

where the second inequality is from Theorem 6 and the third inequality is the generalized mean inequality with $D^+_{0,\Lambda}(\ell(I)^{-1})$ terms. $\square$

We recall for motivation of the hypotheses in the following theorem that the essential α-Hausdorff measure of a set E is $H_\alpha(E)=\inf\{\sum \ell(I_n)^\alpha\colon\ E\subset\bigcup_{n=1}^\infty I_n\cup J,\ |J|=0\}$. The following theorem is related to the essential α-Hausdorff measure of E in Corollary 11.

Theorem 10. *Let $E\subset[0,1]$ have positive measure, $\Lambda=\{<\cdots<\lambda_1<\lambda_0<\lambda_1<\cdots\}\subset\mathbb{R}$ and $0<\alpha<1$. If there exists a sequence of intervals $\{E_n\colon n\in\mathbb{N}\}$ of nonincreasing length, an integer Z and $0<\alpha<1$ such that*

(i) $\bigcup_{n=1}^\infty E_n\supset E$, *and*
(ii) $\sum_{n=1}^Z|E_n|+\sum_{n=Z+1}^\infty|E_n|^\alpha<1$,

and $\dim^+(\Lambda)<1-\alpha$, *then (E,Λ) can be uniformly partitioned into Riesz sequences.*

Proof. Define $F=\bigcup_{n=1}^\infty E_n$, and note that $|F|<|[0,1]|$, and normalize $|[0,1]|=1$. Choose $\epsilon>0$ satisfying

(i) $\frac34+\frac14|F|<1-\epsilon$ and
(ii) $|F|+2\epsilon<\frac12+|F|$.

Choose $M\geqslant Z\in\mathbb{N}$ such that

(i) $|E_M|^{1-\alpha}<\frac{1}{4\sum_{n=M+1}^\infty|E_n|^\alpha}$,
(ii) $4\big(|E_M|^{1-\alpha}+\epsilon\big)\sum_{n=M+1}^\infty|E_n|^\alpha<\epsilon$, and
(iii) $|E_M|^{-1}>R$, where R is chosen from Lemma 8 with $\beta=1-\alpha$ and ϵ as above.

Choose $K\in\mathbb{N}$ such that $M/K<\epsilon$. By Lemma 7, there exists $L\in\mathbb{N}$ such that $|\lambda_j-\lambda_k|>K$ whenever $|j-k|>L$. By Lemma 8, there exists $J\in\mathbb{N}$ such that $D^+_{1-\alpha,\Lambda_i(J)}(r)\leqslant 2R^{\alpha-1}+\epsilon\leqslant 2|E_M|^{1-\alpha}+\epsilon$ for all $r\geqslant|E_M|^{-1}$ and $1\leqslant i\leqslant J$. Finally, let N be the larger of J and L.

Fix $1\leqslant j\leqslant N$, and $\{a_\lambda\colon\ \lambda\in\ell^2(\Lambda_j(N))\}$. We compute

$$\begin{aligned}\sum_{n=1}^\infty\int_{E_n}\Big|\sum_{\lambda\in\Lambda_j(N)}a_\lambda e^{2\pi i\lambda\xi}\Big|^2 &= \sum_{n=1}^M\int_{E_n}\Big|\sum_{\lambda\in\Lambda_j(N)}a_\lambda e^{2\pi i\lambda\xi}\Big|^2\\ &\quad+\sum_{n=M+1}^\infty\int_{E_n}\Big|\sum_{\lambda\in\Lambda_j(N)}a_\lambda e^{2\pi i\lambda\xi}\Big|^2 := S_1+S_2.\end{aligned}$$

Moreover, by Theorem 6 and our choice of K,

$$S_1 \leqslant \left(\sum_{n=1}^{M} \bigl(|E_n| + 1/K\bigr) \right) \sum_{\lambda \in \Lambda_j(N)} |a_\lambda|^2 \leqslant \bigl(|F| + \epsilon\bigr) \sum_{\lambda \in \Lambda_j(N)} |a_\lambda|^2. \tag{3.3}$$

We also have that

$$\begin{aligned} S_2 &\leqslant 2 \sum_{n=M+1}^{\infty} \ell(E_n) D^{+}_{0,\Lambda_j(N)}\bigl(\ell(E_n)^{-1}\bigr) \sum_{\lambda \in \Lambda_j(N)} |a_\lambda|^2 \\ &= 2 \sum_{n=M+1}^{\infty} \ell(E_n)\ell(E_n)^{\alpha-1} D^{+}_{\alpha,\Lambda_j(N)}\bigl(\ell(E_n)^{-1}\bigr) \sum_{\lambda \in \Lambda_j(N)} |a_\lambda|^2 \\ &\leqslant 2 \sum_{n=M+1}^{\infty} \ell(E_n)^{\alpha} \bigl(2|E_M|^{1-\alpha} + \epsilon\bigr) \sum_{\lambda \in \Lambda_j(N)} |a_\lambda|^2 \\ &\leqslant 4\bigl(|E_M|^{1-\alpha} + \epsilon\bigr) \sum_{n=M+1}^{\infty} \ell(E_n)^{\alpha} \sum_{\lambda \in \Lambda_j(N)} |a_\lambda|^2. \end{aligned}$$

In particular,

$$S_2 < \epsilon \sum_{\lambda \in \Lambda_j(N)} |a_\lambda|^2. \tag{3.4}$$

Therefore, combining (3.3) and (3.4), we get that

$$\begin{aligned} \sum_{n=1}^{\infty} \int_{E_n} \Bigl| \sum_{\lambda \in \Lambda_j(N)} a_\lambda e^{2\pi i \lambda \xi} \Bigr|^2 &\leqslant \bigl(|F| + 2\epsilon\bigr) \sum_{\lambda \in \Lambda_j(N)} |a_\lambda|^2 \\ &< \bigl(1 + |F|\bigr)/2 \sum_{\lambda \in \Lambda_j(N)} |a_\lambda|^2. \end{aligned}$$

Therefore, denoting $P = \frac{3+|F|}{4}$, we have

$$\begin{aligned} P \sum_{\lambda \in \Lambda_j(N)} |a_\lambda|^2 &\leqslant (1 - 1/K) \sum_{\lambda \in \Lambda_j(N)} |a_\lambda|^2 \\ &\leqslant \int_{[0,1]} \Bigl| \sum_{\lambda \in \Lambda_j(N)} a_\lambda e^{2\pi i \lambda \xi} \Bigr|^2 d\xi \\ &= \Bigl(\int_{E^c} + \int_{E} \Bigr) \Bigl| \sum_{\lambda \in \Lambda_j(N)} a_\lambda e^{2\pi i \lambda \xi} \Bigr|^2 d\xi \\ &\leqslant \Bigl(\int_{E^c} + \int_{\bigcup E_n} \Bigr) \Bigl| \sum_{\lambda \in \Lambda_j(N)} a_\lambda e^{2\pi i \lambda \xi} \Bigr|^2 d\xi \\ &\leqslant \int_{E^c} \Bigl| \sum_{\lambda \in \Lambda_j(N)} a_\lambda e^{2\pi i \lambda \xi} \Bigr|^2 d\xi + \bigl(1 + |F|\bigr)/2 \sum_{\lambda \in \Lambda_j(N)} |a_\lambda|^2. \end{aligned}$$

It follows that

$$\bigl(1 - |F|\bigr)/4 \sum_{\lambda \in \Lambda_j(N)} |a_\lambda|^2 < \int_{E^c} \Bigl| \sum_{\lambda \in \Lambda_j(N)} a_\lambda e^{2\pi i \lambda \xi} \Bigr|^2,$$

i.e. $\{e^{2\pi i \lambda \xi} 1_{E^c} \colon\ \lambda \in \Lambda\}$ has a lower Riesz bound. Since each $\Lambda_j(N)$ is separated, it also has an upper Riesz bound, which completes the proof. $\square$

Corollary 11. *Let $E \subset [0,1]$ have positive measure, $\Lambda = \{< \cdots < \lambda_1 < \lambda_0 < \lambda_1 < \cdots\} \subset \mathbb{R}$ and $0 < \alpha < 1$. If $H_\alpha(E) < \infty$ and $\dim^{+}(\Lambda) < 1 - \alpha$, then (E, Λ) can be uniformly partitioned into Riesz sequences.*

Proof. If $H_\alpha(E) < \infty$, then we can find intervals $\{E_n\colon n \in \mathbb{N}\}$ such that $E \subset \bigcup_{n=1}^\infty$ and $\sum_{n=1}^\infty |E_n|^\alpha < \infty$.

Choose an integer Z such that

$$\sum_{n=Z}^\infty |E_n|^\alpha < 1 - \alpha.$$

Let

$$F = E \cap \left(\bigcup_{n=Z}^\infty E_n\right).$$

Then, Theorem 10 yields that

$$\{e^{2\pi i\lambda} 1_F\}_{\lambda \in \Lambda}$$

can be uniformly partitioned into Riesz basic sequences given by, say, $\{\Lambda_j\}_{j=1}^K$ with lower Riesz bounds $\{C_j\}_{j=1}^K$.

Now, for any $j = 1, \dots, K$, and any scalars $\{\alpha_\lambda\colon \lambda \in \Lambda_j\}$,

$$\begin{aligned}
\Big\| \sum_{\lambda\in\Lambda_j} a_\lambda e^{2\pi i\lambda x} 1_E \Big\|^2 &= \Big\| \sum_{\lambda\in\Lambda_j} a_\lambda e^{2\pi i\lambda x} 1_F \Big\|^2 + \Big\| \sum_{\lambda\in\Lambda_j} a_\lambda e^{2\pi i\lambda x} 1_{E\setminus F} \Big\|^2 \\
&\geqslant \Big\| \sum_{\lambda\in\Lambda_j} a_\lambda e^{2\pi i\lambda x} 1_F \Big\|^2 \\
&\geqslant C_j \sum_{\lambda\in\Lambda_j} |a_\lambda|^2,
\end{aligned}$$

as desired. □

Example 12. *An example considered by Bourgain–Tzafriri* [2] *and Halpern–Kaftal–Weiss* [11]. Let $\{r_n\colon n \in \mathbb{N}\}$ be a partition of the rational numbers in $[0, 1]$. For each n, let E_n be an interval centered at r_n with length $|E_n| < 2^{-n}$. Let ϕ be the indicator function supported on $E = \bigcup_{n=1}^\infty E_n$. In [11], it was shown that the Laurent operator with symbol ϕ, L_ϕ, cannot be uniformly paved (in fact, something stronger was shown), while in [2], it was shown that L_ϕ can still be paved. In particular, this implies that there is a partition of the integers $\Lambda_1, \dots, \Lambda_N$ such that for each $1 \leqslant i \leqslant N$, $\{1_{E^c} \exp(\lambda)\colon \lambda \in \Lambda_i\}$ is a Riesz sequence. (Note that $(E^c, \mathbb{Z})$ cannot be uniformly partitioned into Riesz sequences.) However, by Theorem 10, whenever $\dim^+(\Lambda) < 1$, (E, Λ) can be uniformly partitioned into Riesz sequences. So, perhaps a better candidate for a counterexample to the Feichtinger conjecture would have been to have a series $\sum a_n$ that converges more slowly to a number less than 1 and to let $|E_n| < a_n$.

We end this paper by presenting an example of a set $E \subset [0, 1]$ and a set $\Lambda \subset \mathbb{Z}$ such that $\dim^+(\Lambda) < 1$ yet (E, Λ) can not be uniformly partitioned into Riesz sequences. We begin by recalling the following theorem.

Theorem 13. *(See [3].) There exists a set $E \subset [0, 1]$ such that whenever $\mathcal{K} \subset \mathbb{Z}$ is such that for all $\delta > 0$, there exist $M, N\ell \in \mathbb{Z}$ such that*

(i) $\ell N^{-1/2} \log^3 N < \delta$, *and*
(ii) $\{M, M + \ell, \dots, M + N\ell\} \subset \mathcal{K}$,

then $\{\exp(\lambda)\colon \lambda \in \mathcal{K}\}$ *is not a Riesz sequence in* $L^2(E)$.

Corollary 14. *There exists a set $E \subset [0, 1]$ such that for each $1 > \beta > 2/3$ there is a set $\Lambda \subset \mathbb{Z}$ such that* $\dim^+(\Lambda) = \beta$ *and* (E, Λ) *cannot be uniformly partitioned into Riesz sequences.*

Proof. Let E be the set guaranteed to exist from Theorem 13. Let $2/3 < \beta < 1$. Let $\gamma = \frac{1-\beta}{\beta}$. Define $\Lambda = \bigcup_{j=1}^\infty \{Q_j + \alpha \lceil j^\gamma \rceil\colon 0 \leqslant \alpha \leqslant j\}$, where the Q_j's are chosen to be some rapidly increasing sequence such as 2^{2^j}. It follows that $\sup_{x\in\mathbb{R}} |\#(\Lambda \cap Q_{j^{\gamma+1}}(x))| \approx j$, and so $\dim^+(\Lambda) = \frac{1}{1+\gamma} = \beta$.

Now, let N be a positive integer and write $\Lambda = \{\lambda_n\colon n \in \mathbb{N}\}$ where $\lambda_i < \lambda_j$ when $i < j$. Let $\Lambda_N = \{\lambda_{Nn}\colon n \in \mathbb{N}\}$. We show that Λ_N satisfies (i) and (ii) of the hypotheses of Theorem 13, which completes the proof of this corollary. Let $\delta > 0$. For each j, let k_j be the smallest nonnegative inteer such that $Q_j + k_j \lceil j^\gamma \rceil \in \Lambda_N$. We have that

$$\{Q_j + (\alpha N + k_j)\lceil j^\gamma \rceil\colon 0 \leqslant \alpha < \lfloor j/N \rfloor\} \subset \Lambda_N.$$

Since $\gamma < 1/2$, we can find j such that

$$\lfloor j/N\rfloor^{-1/2}N\lceil j^{\gamma}\rceil \log^3\lfloor j/N\rfloor < \delta,$$

which finishes the proof. $\square$

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